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SIMPLICAL HOMOLOGY THEORY AND ITS APPLICATION

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Abstract
This paper explores the basic ideas of simplicial structures that lead to simplicial homology theory, and introduces singular homology in order to demonstrate the equivalence of homology groups of homeomorphic topological spaces. It concludes with a proof of the equivalence of simplicial and singular homology groups.

1. Introduction.

Homology theory is essentially an algebraic study of the connectivity properties of a space. The homotopy groups, $\pi_n(Y)$. Although they are appealing intuitively, the homotopy groups are difficult to calculate even for comparatively simple spaces. The simplicial homology groups developed in this paper permit us to answer questions about connectivity similar to those answered by means of homotopy groups. And the simplicial homology groups are computed by almost mechanical methods. On the other hand,the difficulties in homology theory are found in the underlying structures and the combinatorial approach which, for the beginning student, seems to disguise the motivation for an inordinate length of time.

To help the beginner keep sight of the forest, we will discuss at some length the 2-dimensional torus T pictured in Fig. 6-1. Our aim in this discussion is to explain the geometric significance of the purely algebraic concepts to be formulated shortly. First, look at T from the point-set standpoint. Clearly, this surface is a compact, connected and locally Euclidean metric space. It is also locally connected, etc. Of course, all such information above does not characterize the torus. All of these facts are also ture of the 2-dimensional sphere as well. Suppose that our goal is modest, namely, that it is to distinguish topologically between *T* and S^2. How might it be done?

An immediate answer can be given by computing the fundamental groups of *T* and S^2. It turns out that the group $\pi_1(S^2)$ is a trivial group whereas $\pi_1(T)$ is not (such a curve as Z in Fig. 1.1 cannot be shrunk to a point in *T*). Thus we already have knowledge that suffices to distinguish

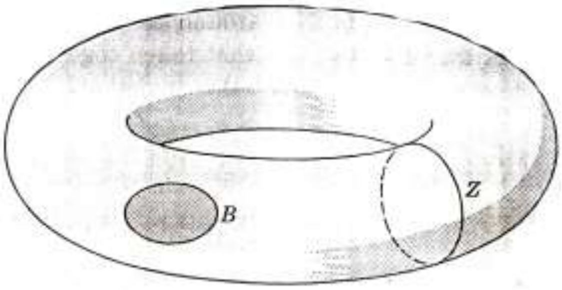

Figure 1.1

Between a tours and a 2-sphere. Let us proceed, however, to give further study to the torus.

Envisioning a 2-sphere, it is intuitively obvious that any closed curve on the surface forms the boundary of a portion of the sphere. Or in equivealent terms, any closed curve on S^2 disconnects S^2. The same is not true of the torus. For cutting along the curve Z in Fig. 1.1 does not disconnect the torus. This implies that the curve Z is not the boundary of a portion of *T*. Of course, there are closed curves, such as B in Fig. 1.1 which are boundaries. The curve B may be considered as the boundary of either the shaded disc or of the complement of that disc in *T*.

Because the intuitive idea of a closed curve includes the notion that it "goes around something" and because it is 1-dimensional, we will temporarily and imprecisely refer to any closed curve such as *B* or *Z* in Fig. 1.1 as a 1-dimensional cycle on *T*. Note that while we have pictured only simple closed curves on *T*, we do not so restrict our cycles. Those special cycles, such as *B*, that bound a portion of the tours *T* do not tell us much about the structure of the torus in the large. We will merely call them bounding 1-cycles and ignore tham. It is the nonbounding 1-cycles, such as Z, that interest us.

There is obviously an uncoundatable number of such nonbounding 1-cycles on the tours. By utilizing simple notions, we will reduce this cardinality drastically. First, the two cycles Z_1 and Z_2 shown in Fig. 1.2 are not

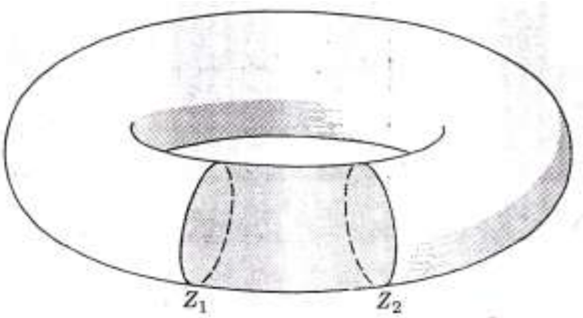

Figure 1.2

intrinsically different since they both go aground the torus once latitudinally. More to the point, however, is the fact that taken together they form the boundary of a portion of the torus (e.g., the shaded cylinder).

2. **Oriented complexes.** As we know from analytic geometry, the concept of a directed (oriented) line segment allows the introduction of algebraic methods into geometry. In an analogous manner, the oriented simplex permits the use of algebraic tools in our study of complexes. We will gain generality by phrasing our definitions in terms of abstract simplicial complexes, but most of our early complex will be taken from the geometric complexes. This is done to attain our double goal of explaining the geometry underlying homology theory while being sufficiently general to permit the necessary extensions later.

An oriented simplex is obtained from an abstract p-simplex

$$\langle v_0 ... v_p \rangle = \sigma^p$$

as follows. We choose some arbitrary fixed ordering of the vertices $v_0, v_1, ..., v_p$. The equivalence class of even permutations of this fixed ordering is the *positively oriented simplex*, which we denote by $+\sigma^p$, and the equivalence class of odd permutations of the chosen ordering is the *negatively oriented simplex*, $-\sigma^p$. For example, if $\langle v_0 v_1 \rangle = +\sigma^1$, then $\langle v_1 v_0 \rangle = -\sigma^1$. For a geometric simplex $s^1 = \langle p_o p_1 \rangle$, orientation is equivalent to a choice of a positive direction on the line segment. Again if we choosen to let $\langle +\sigma^2 \rangle$ represent $\langle v_0 v_2 v_1 \rangle$, then $\langle v_1 v_2 v_0 \rangle$ and $\langle v_2 v_0 v_1 \rangle$ also represent $+\sigma^2$, while $\langle v_1 v_0 v_2 \rangle$, $\langle v_0 v_2 v_1 \rangle$, and $\langle v_2 v_1 v_0 \rangle$, each represents $-\sigma^2$. For a geometric simplex $s^2 = \langle p_0 p_1 p_2 \rangle$, orientation is equivalent to choosing a positive direction of traversing the three 1-faces of s^2. We note that $\langle p_0 p_1 p_2 \rangle$ and $\langle p_1 p_0 p_2 \rangle$ are opposite cyclic orderings of the vertices p_0, p_1, and p_2 and correspond to opposite directions of traversing the boundary of the 2-simplex.

3. **Incidence numbers**. Given an oriented simplical complex *K*, we associate with every pair of simplexes σ^m and σ^{m-1}, which differ in dimension by unity, an incidence number $[\sigma^m, \sigma^{m-1}]$ defined as follows:

$$[\sigma^m, \sigma^{m-1}] = 0 \qquad \text{if } \sigma^{m-1} \text{ is not a cace of } \sigma^m \text{ in } K;$$

$$[\sigma^m, \sigma^{m-1}] = \pm 1 \qquad \text{if } \sigma^{m-1} \text{ is a face of } \sigma^m \text{ in } K.$$

To decide between +1 and —1 in the case where σ^{m-1} is a face of σ^m, we note that if $\sigma^m = \langle v_0 .... v_m \rangle$, then $+\sigma^{m-1} = \pm \langle v_0 ... \hat{v}_i .... v_m \rangle$ (recall that the circumflex accent denotes the omission of the vertex v_i), where the orientation of σ^{m-1} determines the sign. If $+\sigma^{m-1} = + \langle v_0 .... \hat{v}_i .... v_m \rangle$, consider the oriented simplex $\langle v_i v_0 ..... \hat{v}_i ..... v_m \rangle$. This is either

$+\sigma^m$ or $-\sigma^m$; if it is $+\sigma^m$, we take the incidence number $\left[\sigma^m, \sigma^{m-1}\right]$ to be +1, and if $\left\langle v_i v_0 \ldots \hat{v}_i \ldots v_m \right\rangle = -\sigma^m$, we take $\left[\sigma^m, \sigma^{m-1}\right] = -1$. Again, if $\left\langle v_0 v_1 \ldots \hat{v}_i \ldots v_m \right\rangle = -\sigma^{m-1}$, then $\left[\sigma^m, \sigma^{m-1}\right] = -1$ if $\left\langle v_i v_0 \ldots \hat{v}_i \ldots r_m \right\rangle = +\sigma^m$, and $[\sigma^m, \sigma^{m-1}] = +1$ if $\left\langle v_i v_0 \ldots \hat{v}_i \ldots v_m \right\rangle = -\sigma^m$.

If $\left[\sigma^m, \sigma^{m-1}\right] = +1$, then σ^{m-1} is a positively oriented face of σ^m and if the incidence number is negative, then σ^{m-1} is a negatively oriented face of σ^m. The choice of a positive ordering of the vertices of σ^m clearly induces a natural ordering of the vertices in each face of σ^m. Thus an orientation σ^m induces a natural orientation of its faces. The definition above amounts to this: if σ^{m-1} is a face of σ^m then the incidence number $\left[\sigma^m, \sigma^{m-1}\right]$ is positive or negative depending upon whether the chosen orientation of σ^{m-1} agrees or disagrees with the orientation of σ^{m-1} induced by that of σ^m.

EXAMPLE. If $+\sigma^2 = \left\langle v_0 v_1 v_2 \right\rangle$ and $+\sigma^1 = \left\langle v_1 v_2 \right\rangle$, then it is easily verified that $\left[\left\langle v_0 v_1 v_2 \right\rangle, \left\langle v_1 v_2 \right\rangle\right] = +1$. But if $+\sigma^1 = \left\langle v_2 v_1 \right\rangle$, then we have $\left[\left\langle v_0 v_1 v_2 \right\rangle, \left\langle v_2 v_1 \right\rangle\right] = -1$. For, inserting the missing vertex v_0 in front of σ^1, we have $\left\langle v_0 v_1 v_2 \right\rangle = +\sigma^2$ in the first case and $\left\langle v_0 v_1 v_2 \right\rangle = -\sigma^2$ in the second. The reader should work out a number of similar examples for higher-dimensional simplexes.

The oriented simplicial complex *K*, together with the system of incidence number $\left[\sigma^m, \sigma^{m-1}\right]$, constitutes the basic structure supporting a simplical homology theory. We develop this next. First, however, note that for each dimension *m*, we may associate with *K* a matrix $\left([\sigma_i^m, \sigma_j^{m-1}]\right)$ of incidence numbers, where the index *i* runs over all *m*-simplex of *K* and the index *j* runs over all (m-1)-simplexes. A study of this system of incidence matrices would yield the connectivity properties we wish to investigate. This technique was commonly used in the early days of "combinatorial" topology, but we do not develop it. The group-theoretic formulation to be introduced below evolved slowly during the decade 1925-1935 and seems to have been first suggested by E. Noether.

One basic property of the incidence numbers is needed.

THEOREM 3.1 Given any particular simplex σ_0^m of an oriented simplicial complex *K*, the following relationship among the incidence numbers holds:

$$\sum_{i,j}[\sigma_0^m,\sigma_i^{m-1}].\,[\sigma_i^{m-1},\sigma_j^{m-2}]=0.$$

Proof: Every $(m-2)$ simplex $\langle v_0 \ldots \hat{v}_k \ldots \hat{v}_l \ldots v_m \rangle$ in σ^m is a face of exactly two (*m*-1)-faces of σ^m. Hence the sum

$$\sum_{i=0}^{m}\left[\langle v_0 \ldots v_m\rangle,\langle v_0 \ldots \hat{v}_i \ldots v_m\rangle\right].\,\left[\langle v_0 \ldots \hat{v}_i \ldots v_m\rangle,\langle v_0 \ldots \hat{v}_k \ldots \hat{v}_l \ldots v_m\rangle\right]$$

$$=\left[\langle v_0 \ldots v_m\rangle,\langle v_0 \ldots \hat{v}_k \ldots v_m\rangle\right].\,\left[\langle v_0 \ldots \hat{v}_k \ldots v_m\rangle,\langle v_0 \ldots \hat{v}_k \ldots \hat{v}_l \ldots v_m\rangle\right]$$

$$+\left[\langle v_0 \ldots v_m\rangle,\langle v_0 \ldots \hat{v}_l \ldots v_m\rangle\right].\,\left[\langle v_0 \ldots \hat{v}_l \ldots v_m\rangle,\langle v_0 \ldots \hat{v}_k \ldots \hat{v}_l \ldots v_m\rangle\right].$$

There are several cases to be considered. First, if

$$+\langle v_0 \ldots \hat{v}_m\rangle=\langle v_l v_0 \ldots \hat{v}_k \ldots \hat{v}_l \ldots v_m\rangle$$

and

$$+\langle v_0 \ldots v_m\rangle=\langle v_k v_l v_0 \ldots \hat{v}_k \ldots \hat{v}_l \ldots v_m\rangle,$$

then the first term of the above sum is (+1)(+1). Then there are two sub-cases:

(i) If

$$+\langle v_0 \ldots \hat{v}_l \ldots v_m\rangle=\langle v_k v_0 \ldots \hat{v}_k \ldots \hat{v}_l \ldots v_m\rangle,$$

then we have

$$\langle v_l v_k v_0 \ldots \hat{v}_k \ldots \hat{v}_l \ldots v_m\rangle=-\langle v_0 \ldots v_m\rangle$$

and the second term in the sum is (-1) (+1).

(ii) If

$$\langle v_k v_0 \ldots \hat{v}_k \ldots \hat{v}_l \ldots v_m\rangle=-\langle v_0 \ldots \hat{v}_l \ldots v_m\rangle,$$

then the second term in the sum is (+1)(-1).

Thus in either subcase the sum is zero. The remaining cases are handled similarly.

4. Chains, cycles, and groups. Let *K* denote an arbitrary oriented simplicial complex, finite or not, and let *G* denote an arbitrary (additively written) abelian group. (There will be no essential loss of generality if the reader always thinks of the additive group Z of integers whenever we say "arbitrary abelian group.") We make the following definitions. An *m-dimensional chain on the complex K with coefficients in th group G* is a function C_m on the oriented *m*-simplexes of *K* with values in the group G such that if $C_m(+\sigma^m)=g, g$ an element of G, then $C_m(-\sigma^m)=-g$. If *K* is infinite, then $C_m(\sigma^m)=0,$ the identity element of *G*, for all but a finite number of *m*-simplexes of *K*. The collection of all such *m*-dimensional chains on *K* will be denoted by the symbol $C_m(K,G)$.

We introduce an addition of *m*-chains by means of the usual functional addition. That is, we define

$$(C_m^1+C_m^2)\ (\sigma^m)=C_m^1(\sigma^m)+C_m^2(\sigma^m),$$

where the addition on the right is the group operation in *G*.

THEOREM 4.1 Under the operation just defined, $C_m(K,G)$ is an abelian group, the *m-dmensional chain group of K with coefficients in G*.

The reader may prove Theorem 4.1 merely by verifying the axioms for an abelian group.

If the complex *K* has no *m*-simplexes, we take $C_m(K,G)$ to be the trivial group consisting of the identity element 0 alone and write $C_m(K,G)$ = 0.

An *elementary m-chain* on *K* is an *m*-chain C_m such that $c_m(\pm\sigma_0^m)=\pm g_0$ for some particular simplex σ_0^m in *K* and $c_m(\sigma^m)=0$ whenever $\sigma^m \neq \pm\sigma_0^m$. Such an elementary *m*-chain will be denoted by a formal product $g_0.\sigma_0^m$. Then an arbitrary *m*-chain c_m on *K* can be written as a formal linear combination $\sum g_i.\sigma_i^m$, where $g_i = c_m(+\sigma_i^m)$ and all but a finite number of the coefficients g_i are zero. This notation explains the use of the word $coefficient$. Actually, this notation conveniently tabulates the function c_m in such a way that the addition of such functions is the addition of linear combinations. We use this presentation of chains throughout our subsequent development.

THEOREM 4.2 If *K* is a finite complex and α_m is the number of *m*-simplexes in *K*, then the chain group $C_m(K,G)$ is isomorphic to the direct sum of α_m groups, each isomorphic to the coefficient group *G*. If *K* is infinite, the $C_m(K,G)$ is isomorphic to the weak direct sum of infinitely many isomorphic copies of *G*.

***Proof*:** If *K* is finite complex and α_m is the number of *m*-simplexes in *K*, then the chain group $C_m(K,G)$ is isomorphic to the direct sum of α_m groups, each isomorphic to the coefficient group G. infinitely many isomorphic copies of *G*.

***Proof*:** If *K* is finite, then the correspondence

$$\sum_{i=1}^{\alpha m} g_i.\alpha_i^m \leftrightarrow (g_{i,} ..., g_{\alpha_m})$$

is the desired isomorphism, as is readily checked. A similar argument will handle the infinite case, simply recalling the definition of a weak direct sum.

The results describes the chain groups completely, but so far there seems to be little if any geometric meaning in our development. This will be corrected shortly, both by the subsequent definitions and by examples. First, we introduce an algebraic mechanism that corresponds to determining the boundary of a portion of a complex. The *boundary operator* ∂ is defined first on elementary chains by the formula

$$\partial(g_0.\sigma_0^m) = \sum_{\sigma^{m-1}}[\sigma_0^m, \sigma^{m-1}].g_0.\sigma^{m-1},$$

Where $[\sigma_0^m, \sigma^{m-1}]$ is the incidence number. We note that $\partial(g_0.\sigma_0^m)$ is an (*m*-1) chain which has nonzero coefficients only the (m-1) faces of the simplex σ_0^m. The above definition of ∂ is extended linearly to arbitrary *m*-chains by setting

$$\partial\left(\sum_i g_i.\sigma_i^m\right) = \sum_i \partial(g_i.\sigma_i^m).$$

It is easy to see that the boundary of an m-chain is an (m-1) chain which depends only upon the *m*-chain itself and not upon the complex on which the *m*-chain is taken.

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